
Using Fat Tails to Model Gray Swans

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Swans: White, Black, & Gray

- ▶ The Black Swan is a metaphor for a rare event with extreme impact
 - ▶ “All swans white” accepted truth until first black one seen
 - ▶ Example: Stock market crash on 10/19/87
 - ▶ Such events tell us that our models are seriously flawed
 - ▶ Term popularized in books by Nassim Taleb
- ▶ Gray Swans are “events of considerable nature which are far too big for the bell curve, which are predictable, and for which one can take precautions” Benoit Mandelbrot (inventor of fractal geometry)
- ▶ “We seem to have a once-in-a-lifetime crisis every three or four years” Leslie Rahl (founder of Capital Market Risk Advisors)

If the Bell Curve is Such a Bad Model, Why Do We Use It?

- ▶ Central Limit Theorem
 - ▶ Sums of independent & identically distributed (i.i.d.) random variables with finite variance tend towards a normal distribution, regardless of underlying distribution
- ▶ Application to forecasting cumulative wealth
 - ▶ $W_T = W_0(1 + R_1)(1 + R_2)\dots(1 + R_T)$
 - ▶ $\ln W_T = \ln W_0 + \ln(1 + R_1) + \ln(1 + R_2) + \dots + \ln(1 + R_T)$
 - ▶ So if log-returns are i.i.d., with finite variance, log of cumulative wealth tends to a normal distribution & cumulative wealth tends to a lognormal distribution

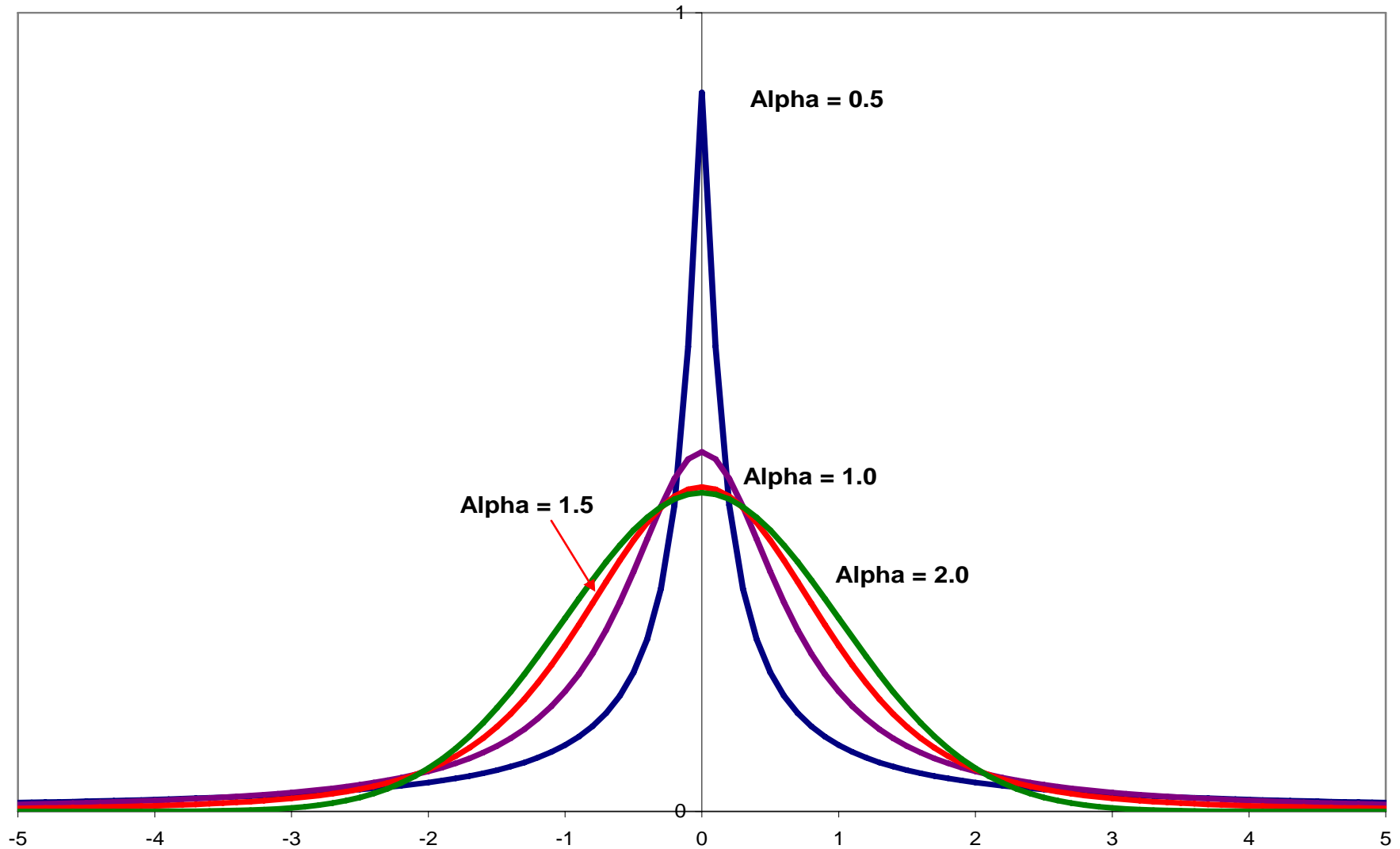
What if Variance is Infinite?

- ▶ In the 1960s, Mandelbrot & his student, Eugene Fama, explored a model in which extreme events occur at realistic frequencies
 - ▶ Log of return relative, $\ln(1 + R)$, has infinite variance
 - ▶ Generalized Central Limit Theorem concludes that log of cumulative wealth has *stable* distribution
 - ▶ Stable distribution have very fat tails
- ▶ Until recently, Mandelbrot-Fama work largely ignored
 - ▶ Hard math
 - ▶ Most portfolio theory does not work with infinite variance
 - ▶ Difficult to estimate model from data

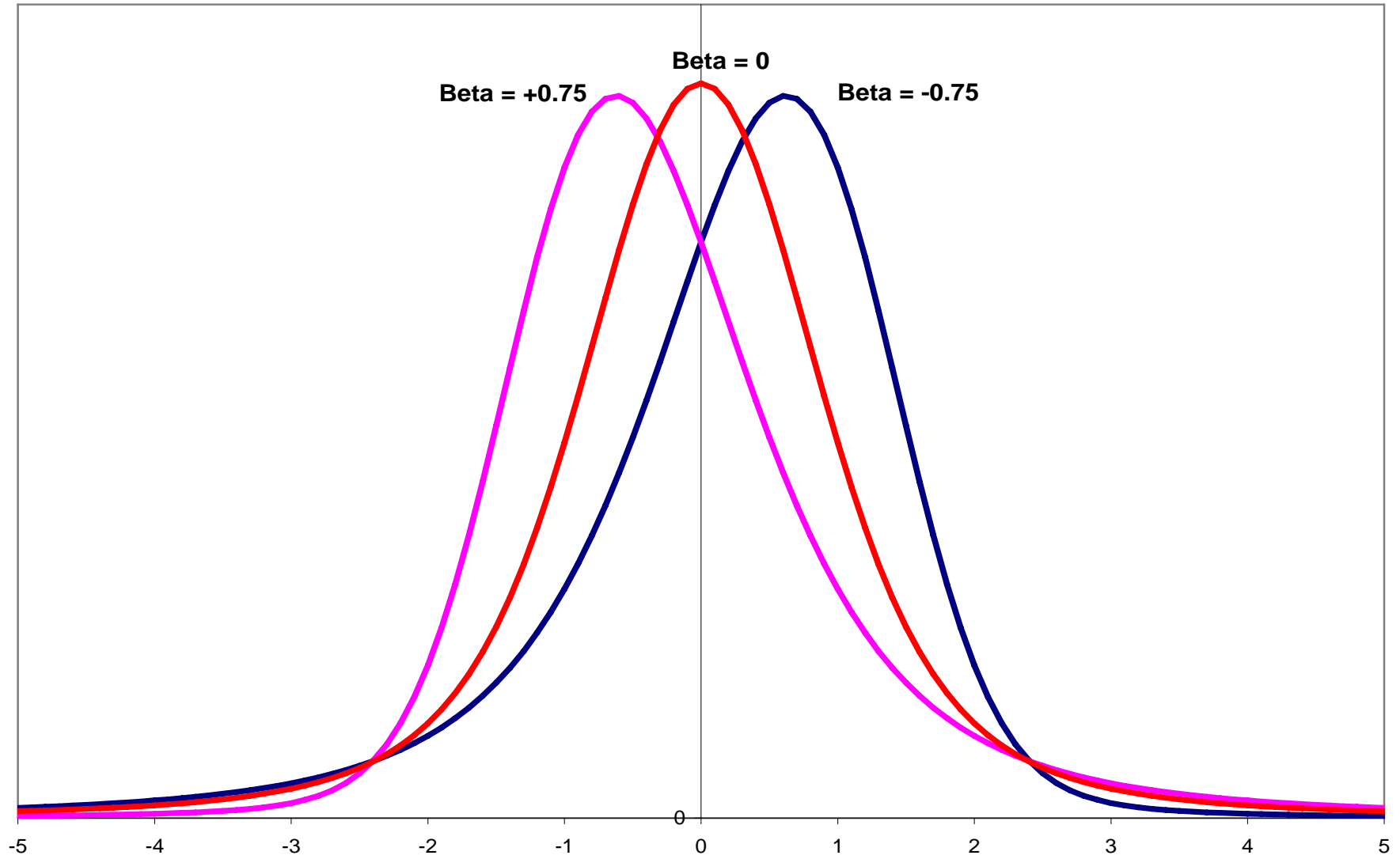
Parameters of Stable Distributions

- ▶ Alpha – Fatness of Tails
 - ▶ $0 < \alpha \leq 2$ (normal)
 - ▶ if $\alpha \leq 1$, mean of distribution infinite
- ▶ Beta – Skewness (if $\alpha < 2$)
 - ▶ (fully left skewed) $-1 \leq \beta \leq 1$ (fully right skewed)
 - ▶ if $\beta = 0$, distribution symmetric
- ▶ Gamma – Scale
 - ▶ $\Gamma > 0$
 - ▶ If $\alpha = 2$, $\Gamma^2 = \text{variance}/2$
- ▶ Delta – Location
 - ▶ if $\alpha > 1$, $\delta = \text{mean of distribution}$

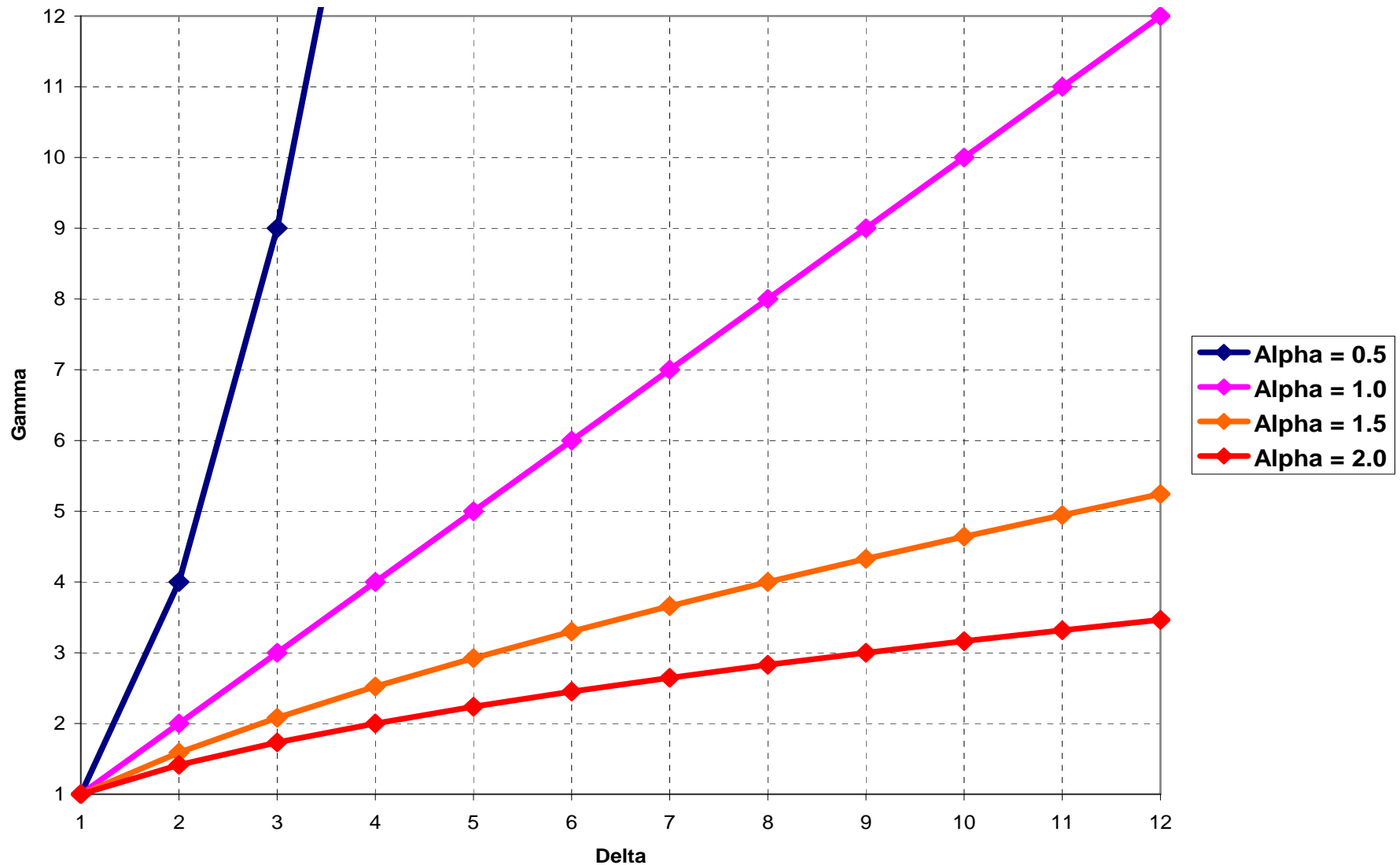
Alpha & The Fatness of Tails



Beta & Skewness



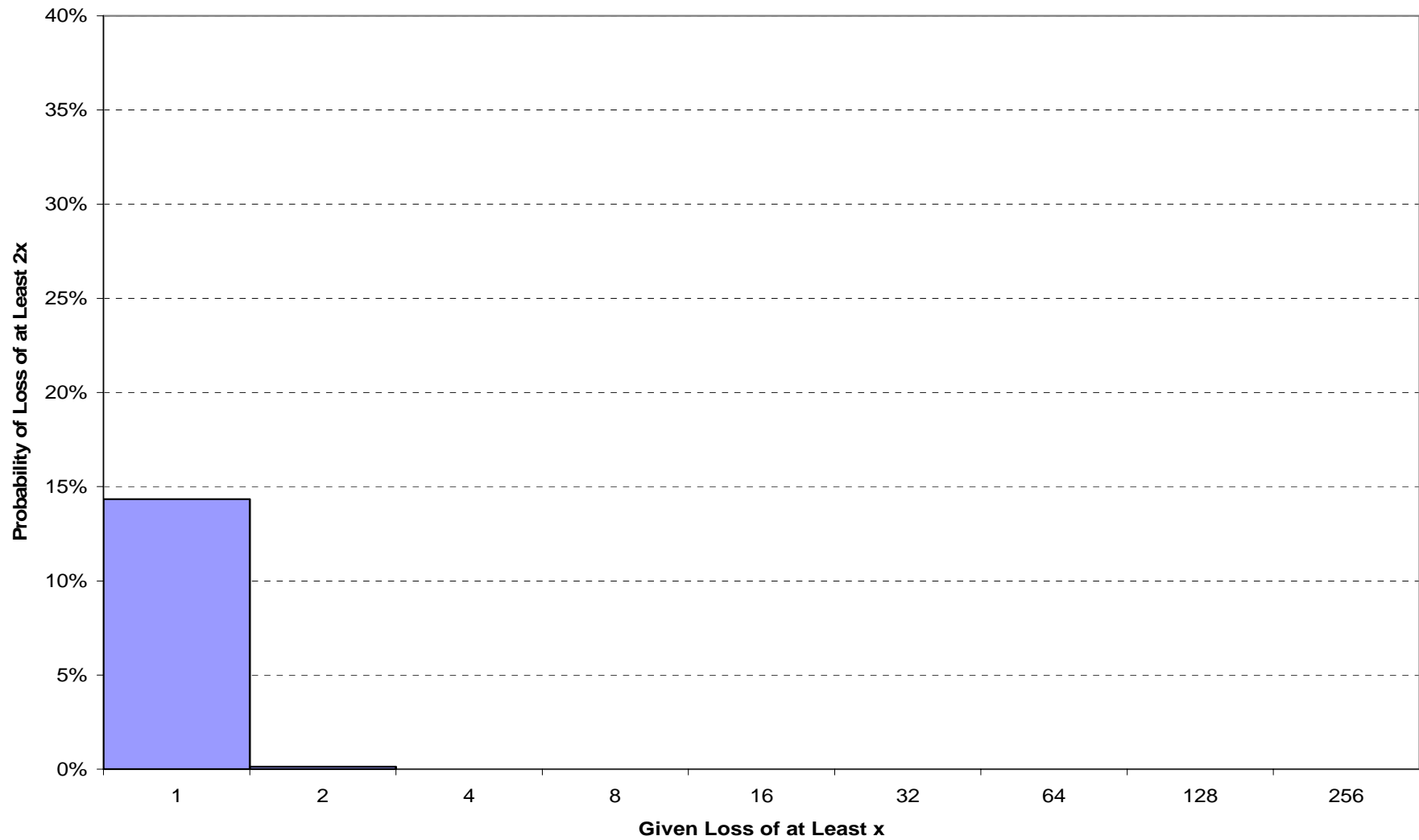
Scaling Property of Gamma & Delta



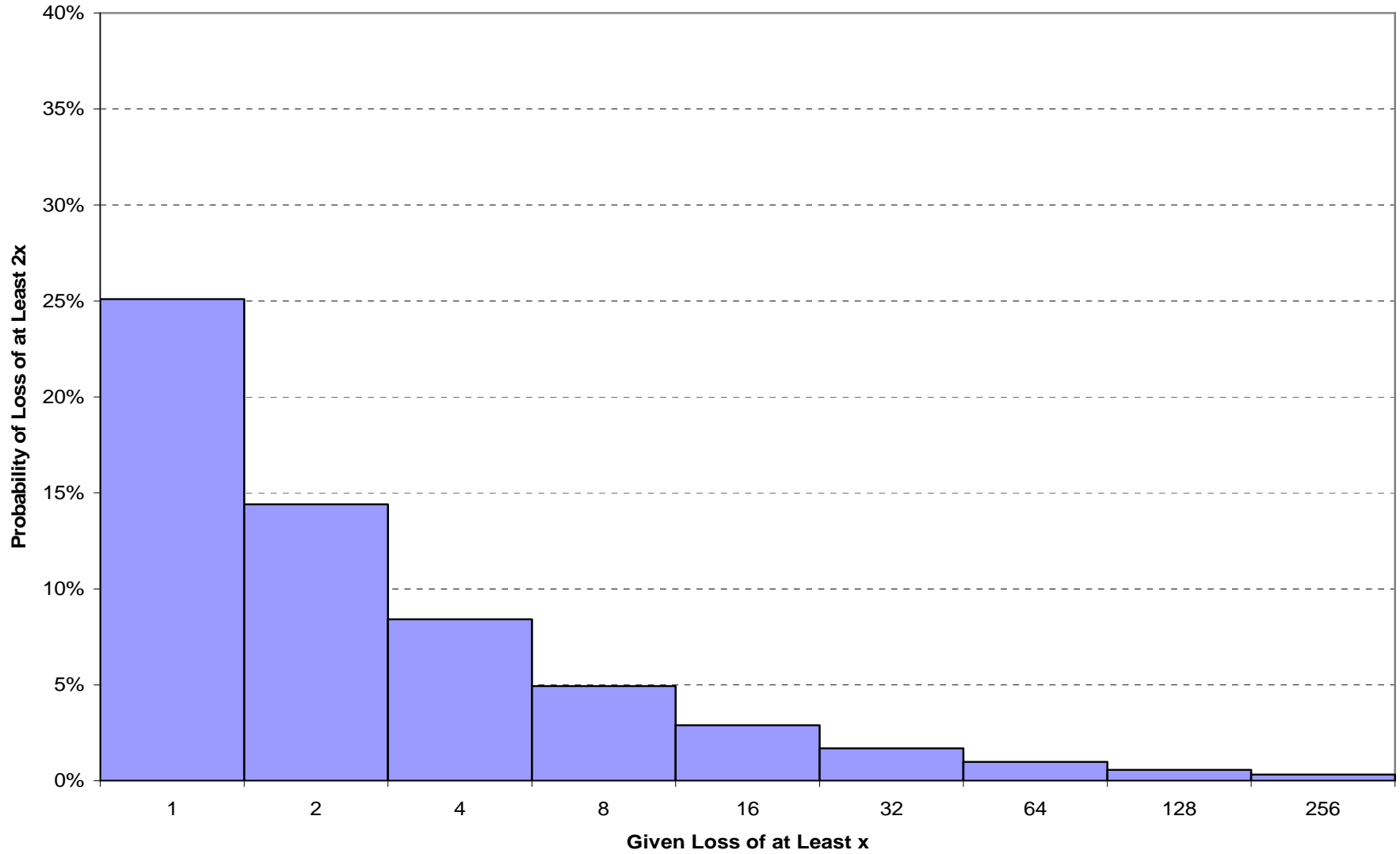
What About Kurtosis?

- ▶ How much of the variance is due to infrequent extreme deviations from the mean, rather than frequent modest deviations
- ▶ Often described as a measure of fat tailness
- ▶ No variance, no kurtosis either
- ▶ Stable distributions have fatter tails than distributions with finite kurtosis

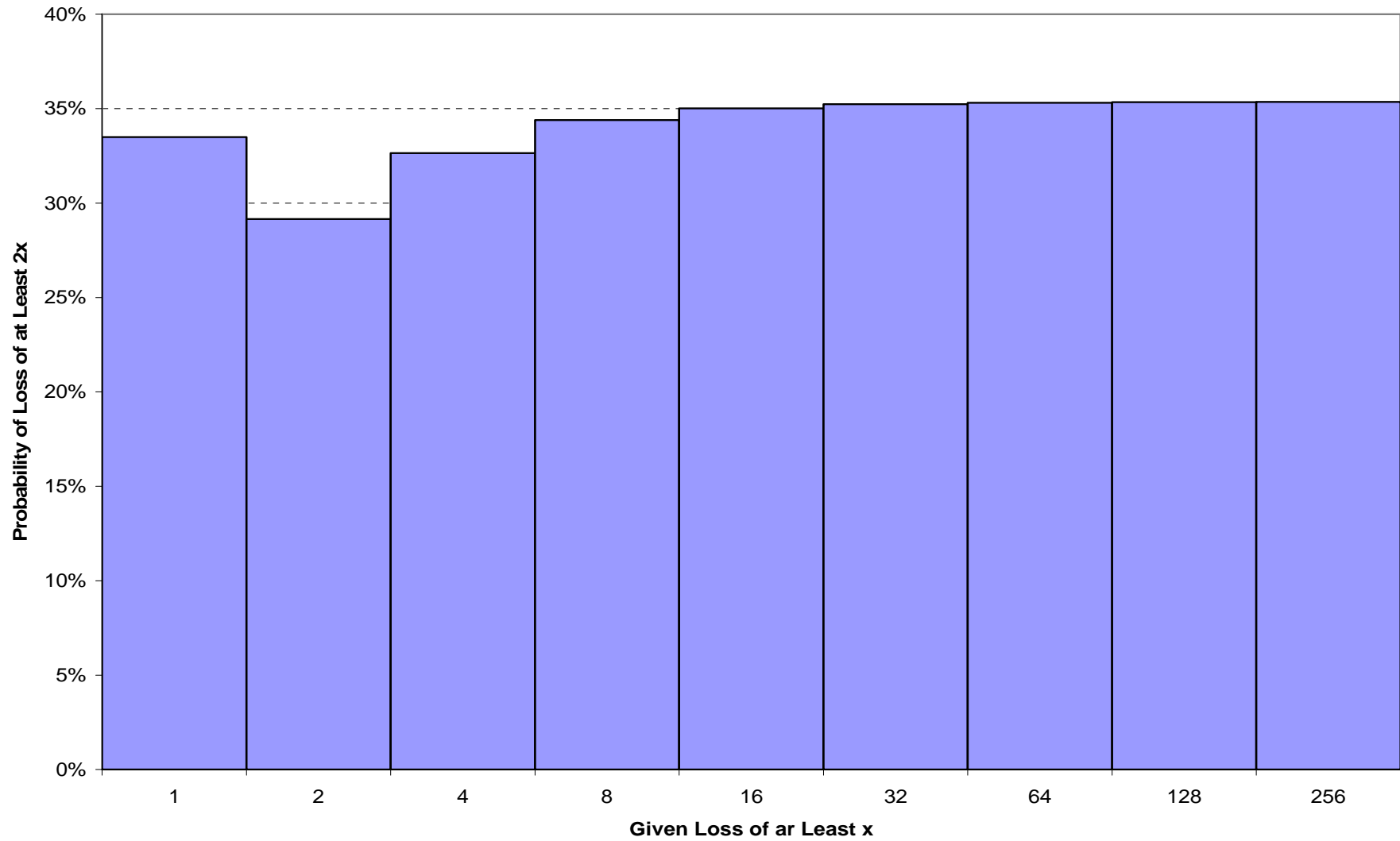
Normal Distributions: Thin Tails



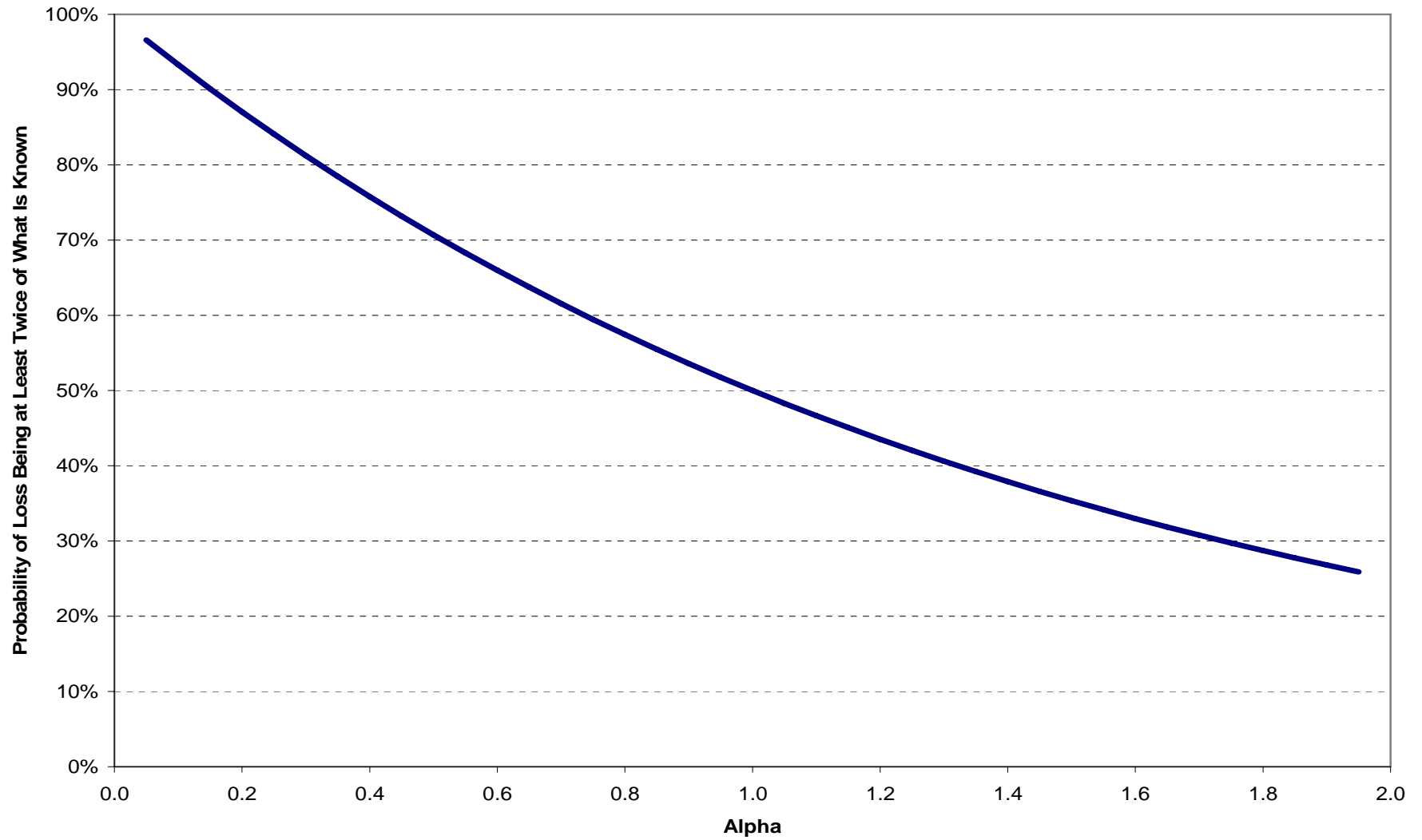
Finite Kurtosis: Dieting Tails



Stable Distributions: Fat Tails



Alpha & Conditional Tail Probability



Log-Stable Model of Returns & Cumulative Wealth

- ▶ Lognormal model: $\ln(1 + R)$ has a normal distribution
- ▶ Log-stable model: $\ln(1 + R)$ has a stable distribution
- ▶ Parameters:

$\ln(1 + R)$		$\ln W_T$	
Normal	Stable	Normal	Stable
2	alpha	2	alpha
-	beta	-	beta
$\sigma/\sqrt{2}$	gamma	$\sqrt{T} \cdot \sigma/\sqrt{2}$	$T^{1/\alpha} \cdot \gamma$
mu	delta	$T \cdot \mu$	$T \cdot \delta$

EnCorr Application: Histogram Overlay in Analyzer

▶ Procedure

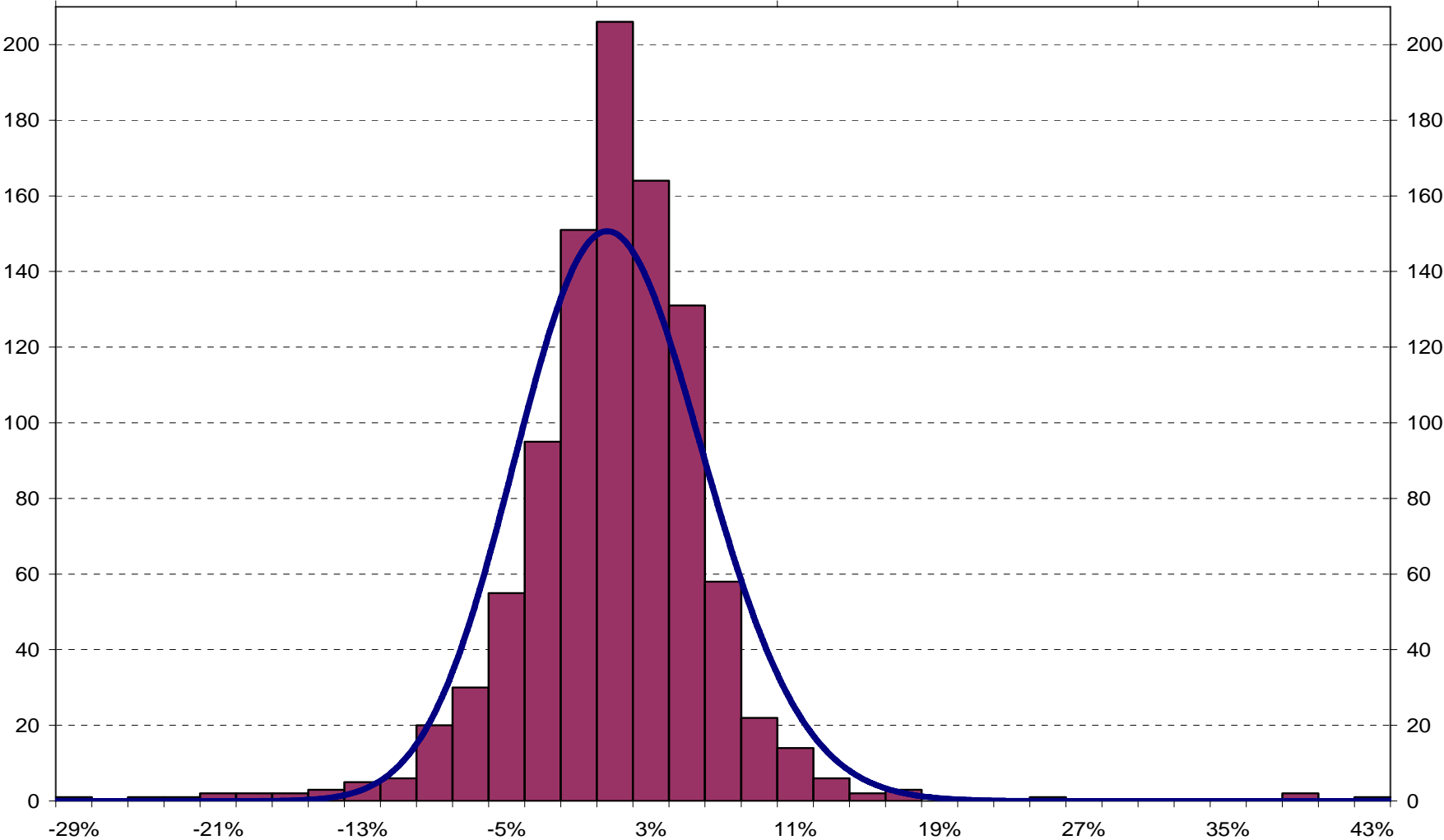
- ▶ Calculate historical log-returns
- ▶ Fit parameters of stable distribution to historical log-returns
- ▶ Draw resulting distribution of returns over histogram

▶ Results

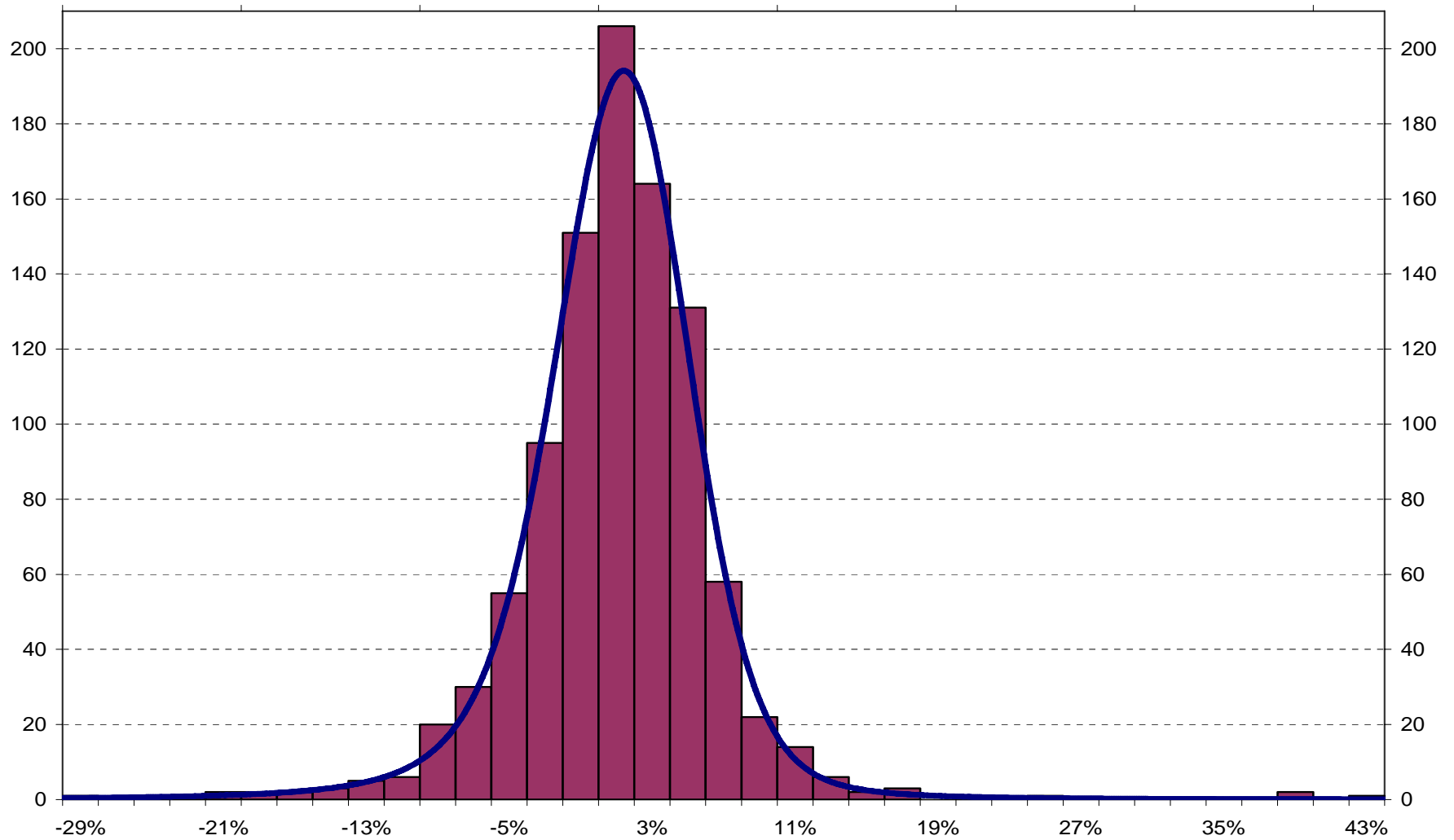
- ▶ Distribution curve that fits data better than lognormal model
- ▶ Example: Monthly returns on S&P 500 Jan. 1926 – Dec. 2007

Parameter	Lognormal	Log-Stable
Alpha	2.000	1.6653
Beta	-	-0.4491
Gamma	0.0366	0.0283
Delta	0.0084	0.0066

Histogram of S&P 500 with Lognormal Overlay



Histogram of S&P 500 with Log-Stable Overlay



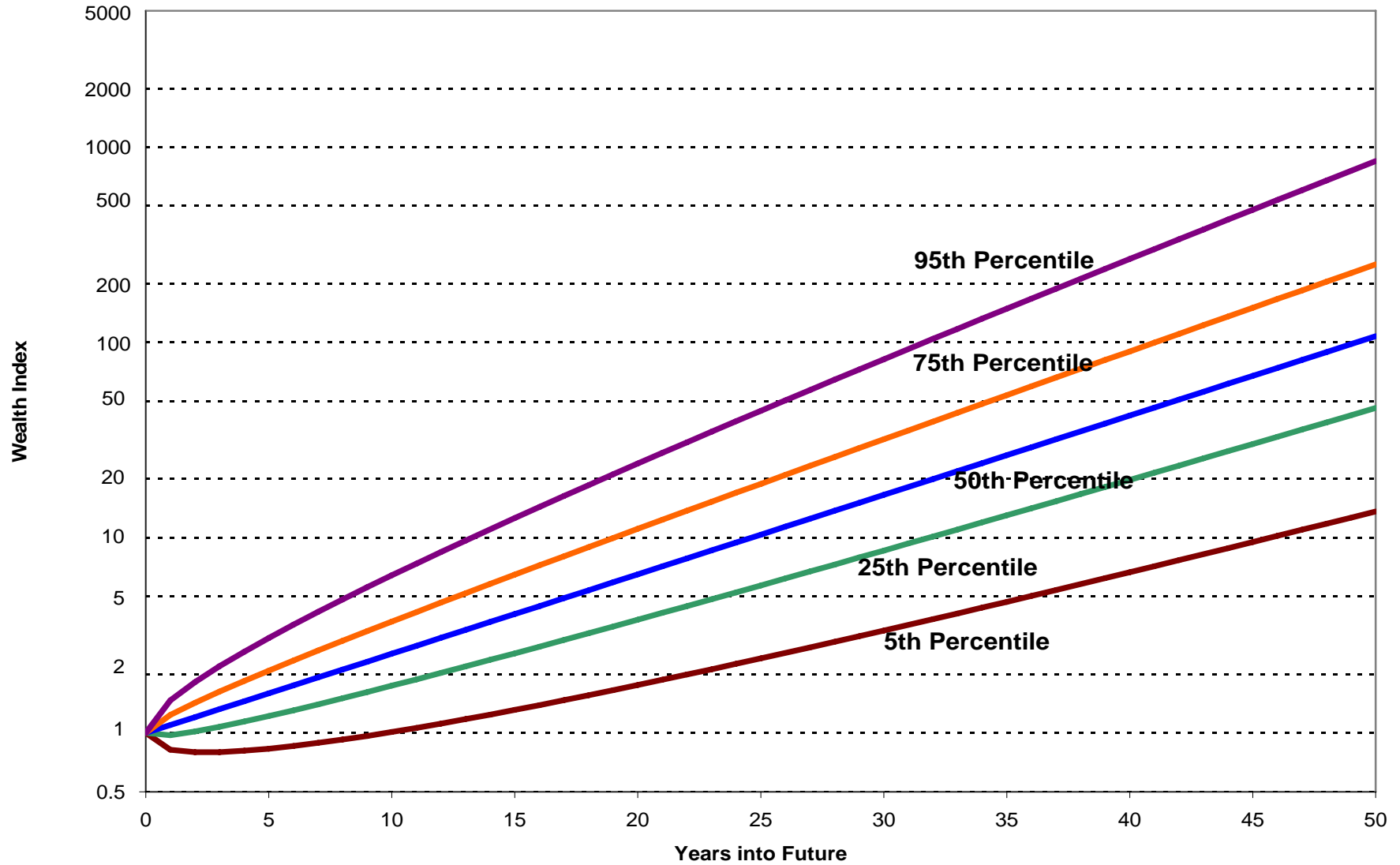
Applications in EnCorr Optimizer: Forecasting & Simulation

- ▶ Use log-stable model in place of lognormal model
- ▶ Setting the four parameters
 - ▶ Use lognormal model value for $E[\ln(1 + R)]$ for delta
 - ▶ Fit parameters of stable distribution to historical log-returns
 - ▶ Use fitted values of alpha & beta
 - ▶ Annualize fitted values of gamma using scaling property

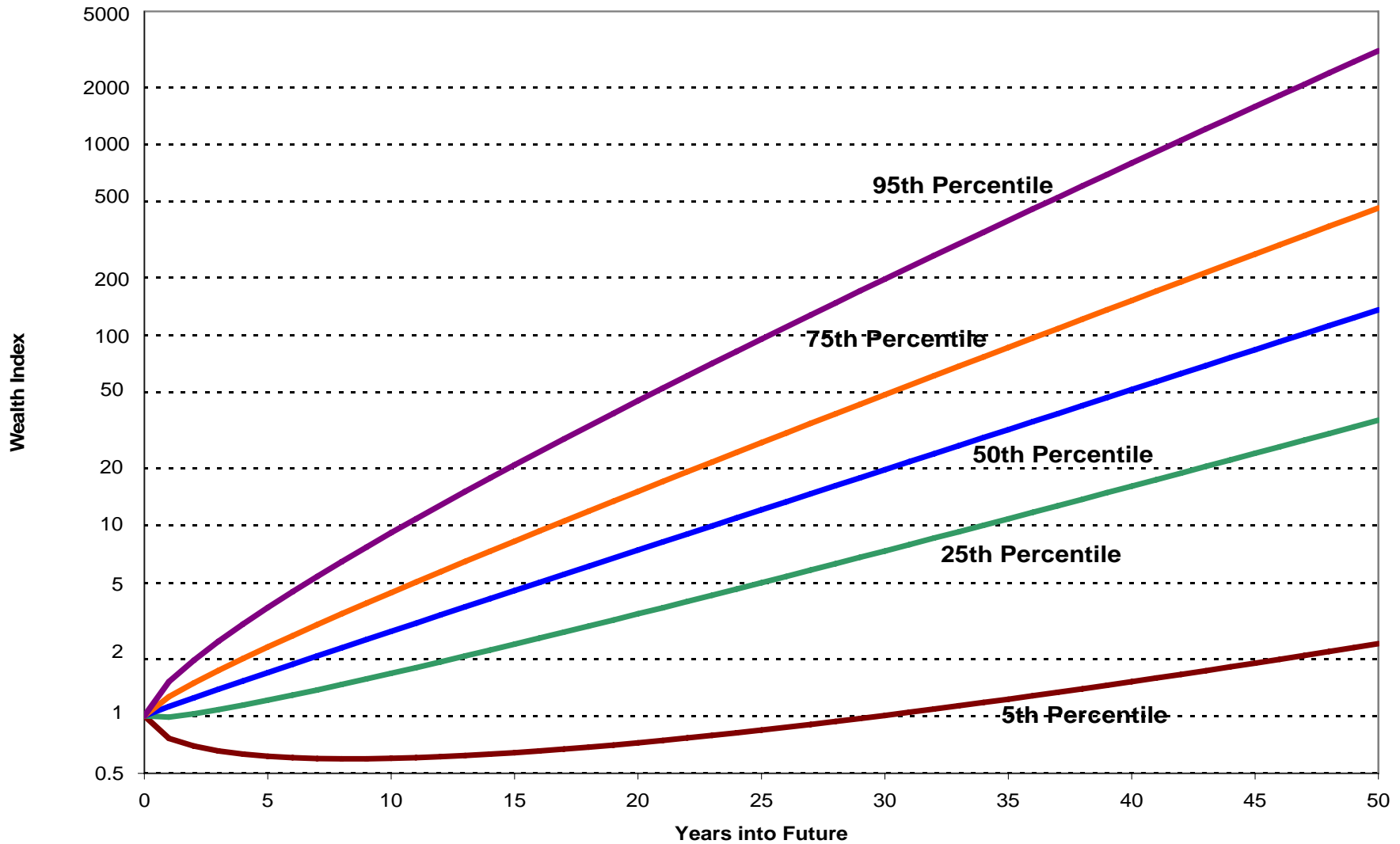
Example: Lognormal Model of S&P 500

- ▶ Set delta using SBBI data
 - ▶ Historical equity premium = 7.05%
 - ▶ Recent Treasury yield = 4.50%
 - ▶ $E[R] = 7.05\% + 4.50\% = 11.56\%$
 - ▶ Historical standard deviation = 19.97%
 - ▶ Lognormal model gives $E[\ln(1 + R)] = 0.0936 = \text{delta}$
- ▶ Set alpha & beta from fit of monthly historical data
 - ▶ Alpha = 1.6653
 - ▶ Beta = -0.4491
- ▶ Set gamma by annualizing monthly fitted value
 - ▶ $\text{Gamma}_A = 12^{1/\alpha} \cdot \text{gamma}_M = 0.1260$

Wealth Forecasting Using Lognormal Model



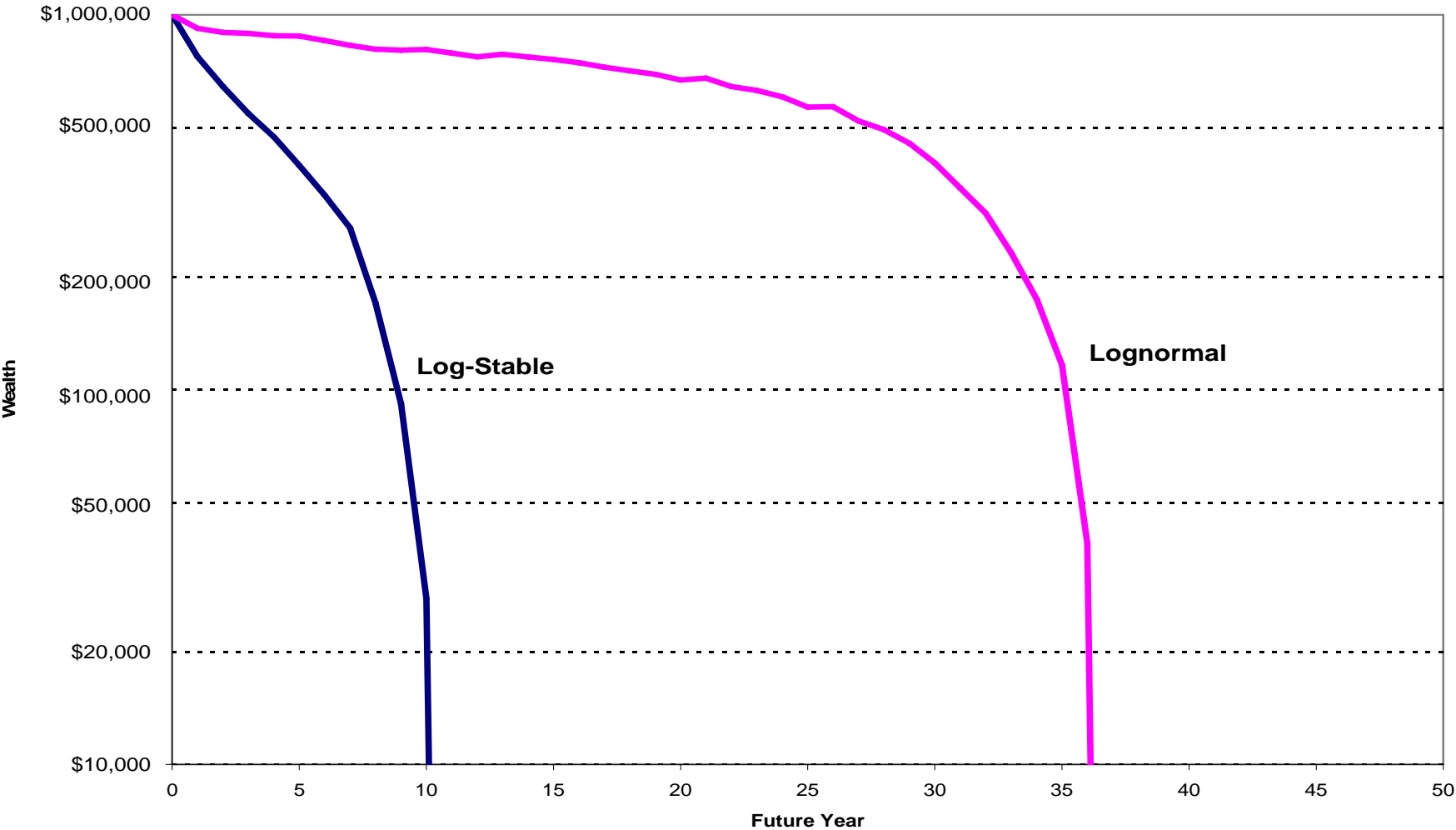
Wealth Forecasting Using Log-Stable Model



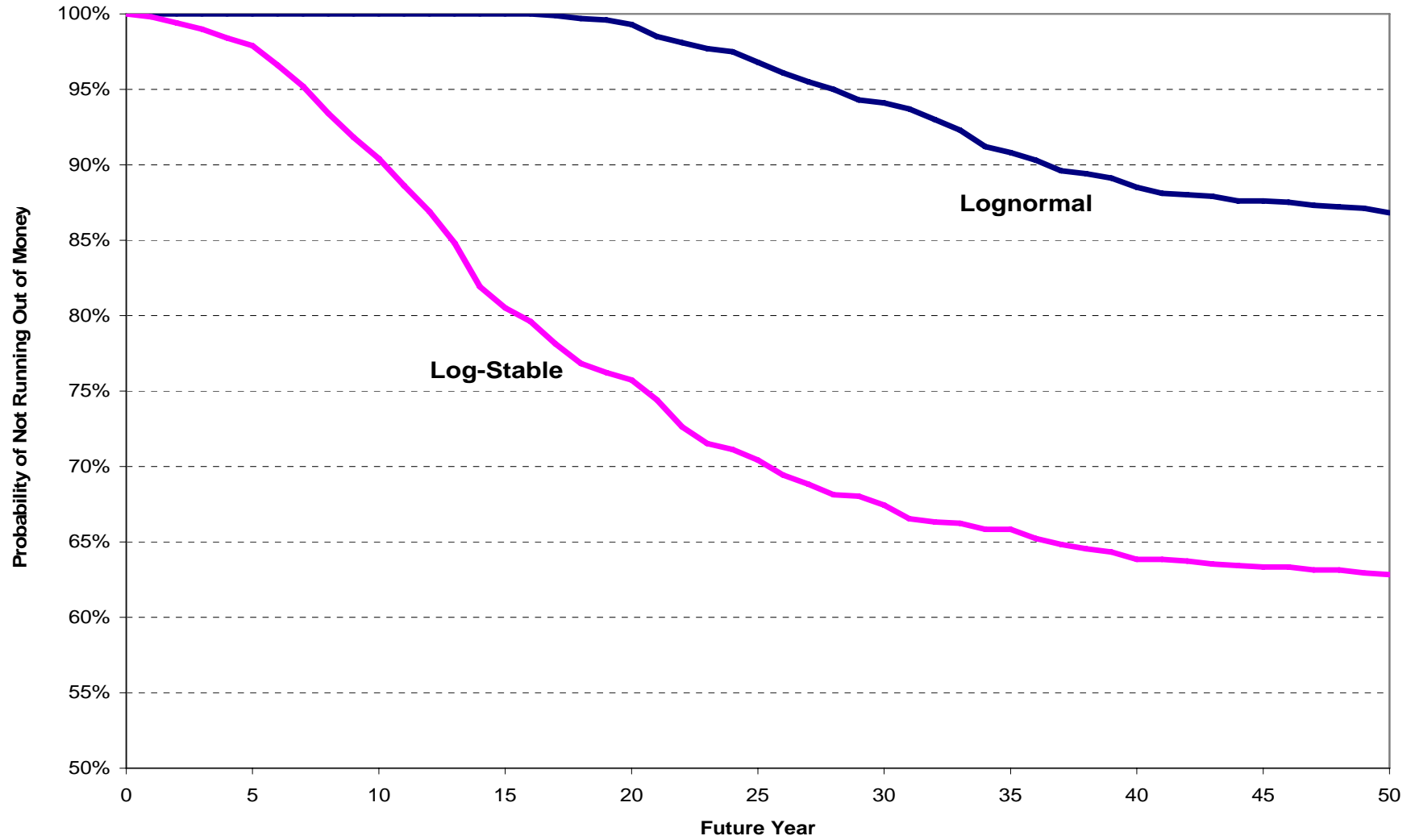
Monte Carlo Simulation with Log-Stable Distributions

- ▶ Example: Drawdown Problem
 - ▶ Start with \$1,000,000
 - ▶ Invest in S&P 500
 - ▶ Assume returns follow log-Stable distribution
 - ▶ Withdraw \$80,000 per year for 50 years
- ▶ Use Monte Carlo Simulation
 - ▶ Run 1,000 simulations of 50 years each
 - ▶ Calculate percentiles of wealth for each year
 - ▶ Calculate probability of not running out of money each year

Simulated Wealth Paths: 10th Percentile



Probability of Not Running Out of Money



Summary

- ▶ Significant market events occur far more frequently than predicted by the lognormal model of returns
- ▶ The basis of the lognormal normal model is the Central Limit Theorem which assumes that variance is finite
- ▶ If variance is infinite, the log-stable model follows from the Generalized Central Limit Theorem
- ▶ Stable distributions have parameters for fatness of tails, skewness, scale, & location
- ▶ The log-stable model will be an alternative to the lognormal model in the EnCorr Analyzer & Optimizer

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